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A STUDY OF THE HYDROMAGNETIC INDUCTION EQUATION

BY

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A Study of the Hydromagnetic Induction Equation

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ABSTRACT

We consider the hydromagnetic induction equation for an unbounded incompressible cosmic fluid of finite conductivity and attempt to find solutions. A partial integration is found possible in a linear velocity shear or when the magnetic Reynolds number is large compared to unity. To complete this integration or to integrate the induction equation when there is no coupling between the components of the magnetic field leads to the "substantial diffusion equation". The integration of this equation is reduced to solving a Fredholm integral equation. Several relations of general interest for a Lagrangian description of fluid flow are obtained.

Contents

I. Introduction

An analysis of dimensional relations in conducting fluids (Elsasser, 1954) shows that Maxwell's equations for a cosmic fluid can be written

$$\nabla \times \boldsymbol{E} = - \frac{\partial \boldsymbol{B}}{\partial t} \tag{1}$$

$$\nabla \times \boldsymbol{B} = \mu \ (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \ .$$

$\boldsymbol{v}$ is the fluid velocity, and the other symbols have their usual meanings. Eliminating between these equations, we obtain the hydromagnetic induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{v} \times \boldsymbol{B} - \nu \ \nabla \times \nabla \times \boldsymbol{B} , \tag{2}$$

where

$$\nu = (\mu \sigma)^{-1} \tag{3}$$

is the "magnetic viscosity." We will assume throughout that the fluid is unbounded, homogeneous, and incompressible. This means that ν is constant and

$$\nabla \cdot \boldsymbol{v} = 0. \tag{4}$$

We will further assume a stationary flow, i.e.

$$\frac{\partial \boldsymbol{v}}{\partial t} = 0, \quad \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}). \tag{5}$$

Our purpose in these pages will be to obtain integrals of the induction equation under these and certain other conditions. The problem will be treated as kinematical, with $\boldsymbol{v}$ given. Unless otherwise specified the coordinate system used is cartesian and defined by the unit base vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Scattered throughout the work are results of general interest for a Lagrangian description of fluid flow. The most important of these are gathered in Appendix I.

On taking the divergence of (2) one obtains, since the divergence of a curl is identically zero,

$$\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{B} = 0 \tag{6}$$

If at any time $\boldsymbol{B}$ vanishes everywhere in space, than at all times

$$\nabla \cdot \boldsymbol{B} = 0$$

The usual proof of (7) given in textbooks on electrodynamics does not include the induction current, $\sigma \mathbf{v} \times \mathbf{B}$. Equation (2) may now be written in the more familiar form

$$\frac{\partial}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} + \nu \nabla^2 \mathbf{B} \, , \tag{8}$$

where use has been made of a familiar vector identity.

In seeking physically significant solutions of the induction equation, the solenoidal character of $\mathbf{B}$ must be kept constantly in mind. While it is true that any solution of (2) is automatically divergence-free, this is not so for solutions of (8). In fact taking the divergence of (8) shows that

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = \nu_m \nabla^2 \nabla \cdot \mathbf{B} \, . \tag{9}$$

Although its solutions are not generally divergence-free their divergence, nevertheless, obeys a diffusion equation, so that if initially divergence-free, they remain so. We conclude that (8) is equivalent to (2) provided (7) is true at t = 0.

II. The Lagrangian viewpoint

When the fluid has infinite conductivity, so that $\nu = 0$, (1-8) may be written

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\cdot\nabla)\mathbf{v}\,, \tag{1}$$

where d/dt denotes the substantial derivative, $\partial/\partial t + \mathbf{v}\cdot\nabla$, of hydrodynamics. In writing (1) we have used the identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} \tag{2}$$

and imposed the condition that $\nabla\cdot\mathbf{B} = 0$. This has the form of the Helmholtz vorticity equation of hydrodynamics, which equation was integrated by Cauchy (Brand, 1947), using the Lagrangian formulation of hydrodynamics. The integral was rediscovered by Lundquist (1951 and 1952) and applied to the hydromagnetic equation (1)

Let $$\mathbf{r} = \mathbf{r}(\mathbf{r}^{o}, t) \tag{3}$$

designate the position of a fluid particle in terms of its initial position, $\mathbf{r}^{o}$, and the time. The Cauchy-Lundquist integral of (1) then is

$$\mathbf{B} = (\mathbf{B}^{o}\cdot\nabla^{o})\,\mathbf{r}\,, \tag{4}$$

where ∇^{o} stands for the operator with components $\partial/\partial x_i^{o}$, and $\mathbf{B}^{o}$ and $\mathbf{B}$ refer to the field measured at a particle when its position is $\mathbf{r}^{o}$ and $\mathbf{r}$, respectively.

The importance of a Lagrangian description of the fluid flow is at once apparent. The particle trajectories are solutions of the differential equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}\,, \tag{5}$$

and the initial position, $\mathbf{r}^{o}$, enters by way of the constants of integration. The solution of this equation has been discussed in a previous report (Skabelund, 1955). We note that if t is held fixed,

$$dx_i = \frac{\partial x_i}{\partial x_j^{o}}\, dx_j^{o}$$

so that

$$dx_1 dx_2 dx_3 = \left|\frac{\partial x_i}{\partial x^o_j}\right| dx^o_1 dx^o_2 dx^o_3 \quad ,$$

in which

$$J = \left|\frac{\partial x_i}{\partial x^o_j}\right|$$

is the Jacobian or functional determinant of the transformation $\mathbf{r}^o \rightarrow \mathbf{r}(\mathbf{r}^o, t)$. But the volume is conserved in an incompressible fluid, hence

$$J = 1 \ . \tag{6}$$

It may be remarked that if J^{-1} is the Jacobian of the inverse transformation, $\mathbf{r} \rightarrow \mathbf{r}^o(\mathbf{r}, t)$ then necessarily $JJ^{-1} = 1$, and hence

$$J^{-1} = \left|\frac{\partial x^o_i}{\partial x_j}\right| = 1 \ . \tag{7}$$

A verification of (4) bears out that one must deal with differential operations cautiously on switching between an Eulerian or field description and a Lagrangian or particle formulation. For example, if $\mathbf{B}$ is expressed in Eulerian terms, i.e. if $\mathbf{r}$ is a field variable, then

$$\frac{d\mathbf{B}}{dt}(\mathbf{r}, t) \text{ means } \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B} \ ; \tag{8}$$

whereas if $\mathbf{B}$ is expressed in Lagrangian form, where $\mathbf{r}$ is a particle trajectory, then $\mathbf{r}^o$ and t are the independent variables, and

$$\frac{d}{dt}\mathbf{B}\left[\mathbf{r}(\mathbf{r}^o, t), t\right] \text{ means } \frac{d\mathbf{B}}{dt}(\mathbf{r}^o, t) = \frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}^o, t) \ . \tag{9}$$

$d\mathbf{B}/dt$ as it appears in (1) refers to the substantial derivative (8); but in verifying (4) one must bear in mind that $\mathbf{B}$ is now expressed in Lagrangian form, and (9) applies. In component notation (4) reads

$$B_i = B^o_j \frac{\partial x_i}{\partial x^o_j} \quad , \tag{4'}$$

so that

$$\frac{dB_i}{dt} = \frac{\partial B_i}{\partial t} = B^o_j \frac{\partial}{\partial t} \frac{\partial x_i}{\partial x^o_j} = B^o_j \frac{\partial v_i}{\partial x^o_j} \quad , \tag{10}$$

which is (1). Note that B_j is independent of t and that $\partial/\partial t$ and $\partial/\partial x^o_j$ commute.

Equation (1) is equivalent to (1-8) with $\nu=0$ only if its solutions are divergence-free, but the remarks previously made concerning the divergence of solutions of (1-8) do not hold for solutions of (1). In fact, speaking of $\mathbf{B}$ as a solution of (1), we have

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) + (\mathbf{V}\cdot\nabla)(\nabla\cdot\mathbf{B}) = 0, \tag{11}$$

rather than $\partial\mathbf{B}/\partial t = 0$, as (1-8) would imply. (11) follows after taking the divergence of (1) and using (2) as well as the identity

$$\nabla\cdot\varphi\mathbf{a} = \nabla\varphi\cdot\mathbf{a} + \varphi\nabla\cdot\mathbf{a}\ . \tag{12}$$

It must be established then that solution (4) is solenoidal. The necessity of doing this has evidently been overlooked in previous accounts of the Cauchy-Lundquist integral.

A well-known interpretation of (1) is that lines-of-force of $\mathbf{B}$ are "embedded" in the fluid. With this geometrical picture in mind it seems clear that no new lines are generated during the course of the fluid's motion, so that div $\mathbf{B}$ if initially zero would remain zero. Another way of looking at the problem is to note that (11) has the form of a first-order wave equation, so that div propogates like a wave in an inhomogeneous $\mathbf{V}$-field (Skabelund, 1955). Thus div $\mathbf{B}$, once zero, shoul· remain zero.

Using (4′) the divergence of (4) is

$$\begin{aligned}\frac{\partial B_i}{\partial x_i} &= \frac{\partial B_j^o}{\partial x_i}\frac{\partial x_i}{\partial x_j^o} + B_j^o\frac{\partial^2 x_i}{\partial x_i\partial x_j^o} \\ &= \frac{\partial B_j^o}{\partial x_j^o} + B_j^o\frac{\partial^2 x_i}{\partial x_i\partial x_j^o}\end{aligned}\ , \tag{13}$$

in which the first term on the right is the initial divergence. In s emi-vector form,

$$\nabla\cdot\mathbf{B} = \nabla^o\cdot\mathbf{B}^o + B_j^o\,\nabla\cdot\frac{\partial\mathbf{r}}{\partial x_j^o}\ . \tag{13'}$$

The quantities $\nabla\cdot\partial\mathbf{r}/\partial x_j^o$, which will also be encountered in the following section, must now be evaluated. It is tempting

to say that
$$\frac{\partial^2 x_i}{\partial x_k\, \partial x_j^o} = \frac{\partial^2 x_i}{\partial x_j^o \partial x_k} = \frac{\partial}{\partial x_j^o} \quad \delta_{ik} = 0.$$

This, however, is not true, for x_k is not an independent variable and $\partial/\partial x_k$ does not commute with $\partial/\partial x_j^o$ ---a fact to figure prominently in the next chapter.

To achieve the desired proof, we begin by writing the identities
$$\delta_{\alpha\beta} = \frac{\partial x_\alpha^o}{\partial x_\beta^o} = \frac{\partial x_\alpha^o}{\partial x_i} \frac{\partial x_i}{\partial x_\beta^o} \quad . \tag{14}$$

Thus for $\beta = 1$, we have
$$\frac{\partial x_1^o}{\partial x_1} \frac{\partial x_1}{\partial x_1^o} + \frac{\partial x_1^o}{\partial x_2} \frac{\partial x_2}{\partial x_1^o} + \frac{\partial x_1^o}{\partial x_3} \frac{\partial x_3}{\partial x_1^o} = 1$$

$$\frac{\partial x_2^o}{\partial x_1} \frac{\partial x_1}{\partial x_1^o} + \frac{\partial x_2^o}{\partial x_2} \frac{\partial x_2}{\partial x_1^o} + \frac{\partial x_2^o}{\partial x_3} \frac{\partial x_3}{\partial x_1^o} = 0 \tag{15}$$

$$\frac{\partial x_3^o}{\partial x_1} \frac{\partial x_1}{\partial x_1^o} + \frac{\partial x_3^o}{\partial x_2} \frac{\partial x_2}{\partial x_1^o} + \frac{\partial x_3^o}{\partial x_3} \frac{\partial x_3}{\partial x_1^o} = 0.$$

Solving for $\partial x_1/\partial x_1^o$ gives

$$J^{-1} \frac{\partial x_1}{\partial x_1^o} = \left[(\nabla x_2^o) \times (\nabla x_3^o) \right]_1 \quad . \tag{16}$$

Solving for $\partial x_2/\partial x_i^o$ and $\partial x_3/\partial x_1^o$ and combining with the above result gives

$$\frac{\partial \mathbf{r}}{\partial x_1^o} = (\nabla x_2^o) \times (\nabla x_3^o) \quad . \tag{17}$$

Repeating these steps for the other two sets allows us to write

$$\frac{\partial \mathbf{r}}{\partial x_\alpha^o} = (\nabla x_\beta^o) \times (\nabla x_\gamma^o) = \nabla \times (x_\beta^o \nabla x_\gamma^o), \tag{18}$$

where (α,β,γ) is a cyclic permutation on (1,2,3). The last form on the right follows because the curl of a gradient is zero and, for any scalar and vector, φ and $\mathbf{a}$,

$$\nabla \times \varphi \mathbf{a} = \nabla\varphi \times \mathbf{a} + \varphi \nabla \times \mathbf{a} \quad , \tag{19}$$

identically. Finally, because the divergence of a curl vanishes, we conclude that

$$\nabla \cdot \frac{\partial \mathbf{r}}{\partial x_\alpha^o} = 0 \ . \tag{20}$$

Substitution of this result into (13′) reveals that

$$\nabla \cdot \mathbf{B} = \nabla^o \cdot \mathbf{B}^o = 0 \quad , \tag{21}$$

the last equality on making the initial field divergence-free. Thie completes the proof that (4) is an integral of the induction equation for infinite conductivity. Clearly (20) is the Lagrangian equivalent of $\nabla \cdot \mathbf{V} = 0$, the Eulerian statement of incompressibility.

III. A Partial Integral of the Induction Equation

a. Physical consideration

In the absence of dissipation we have seen that (2-4) is an integral of the induction equation. The form of (2-4) implies that the field at any time is expressible in terms of its initial configuration and the deformation of the fluid. The physical reasons for this analytical form are clear if we picture the lines-of-force as being distorted when dragged along with the fluid. If there is dissipation we must deal with the equation

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{V} + \nu\nabla^2\mathbf{B}$$

$$\text{or } \frac{d\mathbf{B}}{dt} = (\mathbf{B}\cdot\nabla)\mathbf{V} + \nu\nabla^2\mathbf{B}, \qquad (1)$$

which is equivalent to (1-2) or (1-8) provided its solution is divergence-free.

The preceding suggests that (1) might have an integral of the form

$$\mathbf{B} = (\hat{\mathbf{B}}\cdot\nabla^{\circ})\mathbf{r}, \qquad (2)$$

where $\hat{\mathbf{B}}$ is a function depending somehow on $\mathbf{r}$ and t. The presence of a dissipative term in (1) means the lines-of-force now slip relative to the fluid; $\mathbf{B}$, then, is no longer expressible solely in terms of its initial configuration and the fluid deformation, and $\hat{\mathbf{B}}$, unlike $\mathbf{B}^{\circ} = \mathbf{B}(\mathbf{r}^{\circ},0)$ must have the more general form $\hat{\mathbf{B}}(\mathbf{r},t)$. We expect $\hat{\mathbf{B}}$ to obey a diffusion-like equation---one which will reduce $\hat{\mathbf{B}}$ to $\mathbf{B}(\mathbf{r}^{\circ},0)$ when $\nu=0$. Such an equation is

$$\frac{d\hat{\mathbf{B}}}{dt} = \nu\nabla^2\hat{\mathbf{B}}\ ; \qquad (3)$$

for when $\nu=0$, $d\hat{\mathbf{B}}/dt = 0$, meaning that $\hat{\mathbf{B}}$ is constant as one moves with a fluid particle along a stream line, and hence $\hat{\mathbf{B}} = \hat{\mathbf{B}}(\mathbf{r}^{\circ})$. That a solution (2 and 3) exists in certain cases will now be verified.

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b. The conditions for a particular integral

Assuming (2) we have

$$\frac{dB_i}{dt} = \frac{d\hat{B}_j}{dt}\frac{\partial x_i}{\partial x_j^o} + \hat{B}_j \frac{d}{dt}\frac{\partial x_i}{\partial x_j^o}, \tag{4}$$

but the $\partial x_i/\partial x_j$ are expressed in Lagrangian terms so that

$$\frac{d}{dt}\frac{\partial x_i}{\partial x_j^o} = \frac{\partial}{\partial t}\frac{\partial x_i}{\partial x_j^o} = \frac{\partial v_i}{\partial x_j^o} \tag{5}$$

Thus (4) becomes

$$\frac{dB_i}{dt} = \frac{d\hat{B}_j}{dt}\frac{\partial x_i}{\partial x_j^o} + \hat{B}_j \frac{\partial v_i}{\partial x_j^o} \tag{6}$$

or in vector form

$$\frac{d\mathbf{B}}{dt} = \left(\frac{d\hat{\mathbf{B}}}{dt}\cdot\nabla^o\right)\mathbf{r} + (\hat{\mathbf{B}}\cdot\nabla^o)\mathbf{v}. \tag{6'}$$

Consider the quantity $(\mathbf{B}\cdot\nabla)\mathbf{v}$; in component form we have, using (2),

$$B_j \frac{\partial v_i}{\partial x_j} = \hat{B}_k \frac{\partial x_j}{\partial x_k^o}\frac{\partial v_i}{\partial x_j} = B_k \frac{\partial v_i}{\partial x_k^o} \tag{7}$$

or

$$(\mathbf{B}\cdot\nabla)\mathbf{v} = (\hat{\mathbf{B}}\cdot\nabla^o)\mathbf{v}. \tag{7'}$$

Since $\hat{\mathbf{B}}$ is a solution of (3) by assumption, (6') may be written

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\cdot\nabla)\mathbf{v} + \nu(\nabla^2\hat{\mathbf{B}}\cdot\nabla^o)\mathbf{r}, \tag{8}$$

which has the form (1) except for the last term.

It remains to examine $\nabla^2\mathbf{B}$, subject to (2). One obtains

$$\frac{\partial^2 B_i}{\partial x_j \partial x_j} = \frac{\partial^2 \hat{B}_k}{\partial x_j \partial x_j}\frac{\partial x_i}{\partial x_k^o} + 2\frac{\partial \hat{B}_k}{\partial x_j}\frac{\partial^2 x_i}{\partial x_j \partial x_k^o} + B_k \frac{\partial^3 x_i}{\partial x_j \partial x_j \partial x_k^o} \tag{9}$$

or

$$\nabla^2\mathbf{B} = (\nabla^2\hat{\mathbf{B}}\cdot\nabla^o)\mathbf{r} + 2\mathbf{A} + \mathbf{C} \tag{9'}$$

where and have components

$$A_i = \frac{\partial \hat{B}_k}{\partial x_j}\frac{\partial^2 x_i}{\partial x_j \partial x_k^o}, \quad C_i = \hat{B}_k \frac{\partial^3 x_i}{\partial x_j \partial x_j \partial x_k^o}$$

Finally, then, (2) is an integral of (1) with $\hat{\mathbf{B}}$ subject to (3) if and only if

$$\mathbf{A} = \mathbf{C} = 0 \quad . \tag{11}$$

Since the partials with respect to the x_j and x_k^o which appear in A_i and C_i do not commute, the conditions (11) will not, in general, be met. In fact, these conditions will be found to be very restrictive. This is not surprising, for according to (2), (9′), and (11), the quantity $\nabla^2 \mathbf{B}$, which accounts for the diffusion of $\mathbf{B}$, will evolve in the same way as $\mathbf{B}$ itself. Obviously, not many types of flow patterns will permit **this.**

It will be shown presently, however, that (11) can be satisfied under certain conditions. Accepting this fact for the moment, let us see what has been accomplished. The task of integrating (1) has been reduced to that of integrating the auxiliary equation (3). The net effect of our efforts has been to eliminate the term $(\mathbf{B} \cdot \nabla)\mathbf{v}$ from (1). This is a considerable simplification, for in (1) the components of $\mathbf{B}$ are coupled through the term $(\mathbf{B} \cdot \nabla)\mathbf{v}$. The set (1) of three simultaneous equations in all three B_i has been reduced to the three independent equations of (3), in which the B_i are not coupled. In this **respect** (2) is a <u>partial integral</u> of the induction equation. More precisely, this is so provided $\mathbf{B}$, thus determined, has zero divergence.

From (2) we have

$$\frac{\partial B_i}{\partial x_i} = \frac{\partial \hat{B}_j}{\partial x_i} \frac{\partial x_i}{\partial x_j^o} + \hat{B}_j \frac{\partial^2 x_i}{\partial x_i \partial x_j^o} \; ; \tag{12}$$

however, it was established previously that, for an incompressible flow,

$$\nabla \cdot \frac{\partial \mathbf{r}}{\partial x_j^o} = 0,$$

and hence,

$$\nabla \cdot \mathbf{B} = \nabla^o \cdot \hat{\mathbf{B}} \; . \tag{13}$$

It can be shown that for the types of fluid flow which satisfy our criterion (11)

$$\nabla^o \cdot \mathbf{B} = 0 \text{ if } \nabla^o \cdot \hat{\mathbf{B}}(\mathbf{r}^o, 0) = 0,$$

and thus

$$\nabla \cdot \mathbf{B} = 0 \text{ if } (\nabla \cdot \hat{\mathbf{B}})_{t=0} = 0. \tag{14}$$

This demonstration must await our discussion of equation (11) of sec. 4; however, rather than interrupt the argument there, the proof is given in Appendix IV B.

As to the physical significance of $\hat{\mathbf{B}}$, note from (2) that for t = 0

$$\mathbf{B}(\mathbf{r}^{\circ},0) = \hat{\mathbf{B}}(\mathbf{r}^{\circ},0) \ ; \tag{15}$$

thus $\hat{\mathbf{B}}(\mathbf{r},0)$ is the initial value of the field, $\mathbf{B}$. The equation (3) governing $\mathbf{B}$ is a diffusion equation for an observer moving with the fluid. It may also be written

$$\frac{\partial \hat{\mathbf{B}}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \hat{\mathbf{B}} = \nu\, \nabla^2\, \hat{\mathbf{B}}, \tag{3^1}$$

which has the form of a wave equation with dissipation (cf. sec. 4 and Tech. Report 17 of the author). It describes a $\hat{\mathbf{B}}$-field propagating in an inhomogeneous anisotropic $\mathbf{v}$-field and at the same time diffusing. Its solution will be the subject of sec. 4.

c. Satisfying the conditions - Linear velocity shear

Suppose that the streamlines are straight; we may without loss of generality take them to be directed along the 2-axis. It will further be supposed that $\mathbf{v}$ does not vary in this direction, i.e., we assume

$$\mathbf{v} = v(x_1,x_3)\, \boldsymbol{e}_2 \ . \tag{16}$$

The Lagrangian trajectories are given by

$$\frac{dx_1}{dt} = 0, \quad \frac{dx_2}{dt} = v, \quad \frac{dx_3}{dt} = 0, \tag{17}$$

whose solutions are

$$x_1 = x_1^{\circ}, \ x_2 = x_2^{\circ} + vt, \ x_3 = x_3^{\circ} \ . \tag{18}$$

We have seen in the preceding section that a partial integral of the form (2) exists only when the quantities of (10) vanish. If we require the vanishing of these quantities without placing any restrictions on the magnetic field, a necessary and sufficient condition is that

$$\frac{\partial^2 x_1}{\partial x_j \partial x_k^o} = 0. \tag{19}$$

By referring to (18) we see at once that

$$\frac{\partial^2 x_1}{\partial x_j \partial x_k^o} = \frac{\partial^2 x_2}{\partial x_j \partial x_k^o} = \frac{\partial^2 x_3}{\partial x_j \partial x_k^o} = 0 ; \tag{20}$$

It remains only to consider $\partial^2 x_2/\partial x_j \partial x_1^o$ and $\partial^2 x_2/\partial x_j \partial x_3^o$. The first of these can be written

$$\frac{\partial^2 x_2}{\partial x_j \partial x_1^o} = \frac{\partial x_\ell^o}{\partial x_j} \quad \frac{\partial^2 x^2}{\partial x_j \partial x_\ell^o} \tag{21}$$

Now, the $\partial x_\ell^o/\partial x_j$ may be found at once from (18). In more general cases, solving the Lagrangian trajectories for the inverse functions $x_i^o = x_i^o(x_j,t)$ is no easy matter; still the $\partial x_\ell^o/\partial x_j$ can easily be found in terms of the $\partial x_\alpha/\partial x_\beta^o$, once the latter are known---as they will be from the trajectories. This procedure will now be discussed because of its importance to a general Lagrangian formulation, although as mentioned, it is not essential here.

Return to equations (2-14) of sec. 2; if they are viewed this time as comprising three sets of algebraic equations for the nine $\partial x_\alpha^o/\partial x_\beta$ in terms of the $\partial x_\alpha/\partial x_\beta^o$, then we have, on exchanging the roles of $\boldsymbol{r}$ and $\boldsymbol{r}^o$ in (2-17),

$$\frac{\partial \boldsymbol{r}^o}{\partial x} = (\nabla^o x_\beta) \times (\nabla^o x_\gamma), \tag{22}$$

where (α,β,γ) is again a cyclic permutation on (1,2,3).

Using the above result, or proceding directly from (18), shows that all the $\partial^2 x_i/\partial x_j \partial x_k^o$ are zero save four:

$$\begin{aligned} &\frac{\partial^2 x_2}{\partial x_1 \partial x_1^o} = t\,\frac{\partial^2 v}{\partial (x_1^o)^2}, \quad \frac{\partial^2 x_2}{\partial x_3 \partial x_3^o} = t\,\frac{\partial^2 v}{\partial (x_3^o)^2} \\ &\frac{\partial^2 x_2}{\partial x_3 \partial x_1^o} = \frac{\partial^2 x_2}{\partial x_1 \partial x_3^o} = t\,\frac{\partial^2 v}{\partial x_1^o \partial x_3^o} \end{aligned} \tag{23}$$

The restrictions imposed on the flow by (19) are that

$$\frac{\partial^2 v}{\partial (x_1^o)^2} = \frac{\partial^2 v}{\partial x_1^o \partial x_3^o} = \frac{\partial^2 v}{\partial (x_3^o)^2} = 0 \quad , \tag{24}$$

which have a solution

$$v = ax_1 + bx_3 + c \quad . \tag{25}$$

The equi-velocity surfaces are planes $ax_1^o + bx_3^o + c = \text{const.}$, parallel to the streamlines. Thus v must change linearly in a direction normal to the flow (the 2-axis). We could just as well take this direction to be the 1- or 3- axis so that, for example,

$$\mathbf{v} = (ax_3 + b)\,\boldsymbol{e}_2 \quad , \tag{26}$$

which represents a linear velocity shear in the 3-direction. Many velocity fields will obey this condition locally, except at points where $|\mathbf{v}|$ has extrema. Recall that $\mathbf{B}$ has not been limited in any way.

d. Approximate theory for large magnetic Reynolds numbers

The possibility of satisfying the conditions (11) approxmately will now be considered. Suppose the larges of the $\partial x_i/\partial x_k^o$ has an order of magnitude δ; we write

$$\frac{\partial x_i}{\partial x_k^o} \lesssim \delta \quad . \tag{27}$$

Then, if λ is the scale of the fluid motion and if the motion is reasonably "smooth",

$$\frac{\partial^2 x_i}{\partial x_j \partial x_k^o} \lesssim \frac{\delta}{\lambda} \quad \text{and} \quad \frac{\partial^3 x_i}{\partial x_j \partial x_j \partial x_k^o} \lesssim \frac{\delta}{\lambda^2} \quad , \tag{28}$$

whence

$$C = |\mathbf{C}| \sim \hat{B}\,\frac{\delta}{\lambda^2} \quad , \tag{29}$$

We need next to determine the order of the $\partial\hat{B}_k/\partial x_j$; it is <u>not</u> $\hat{B}/\lambda$, for the $\hat{\mathbf{B}}$-field has not necessarily diffused through a distance characterized by λ. The ordinary diffusion equation, $\partial\mathbf{B}/\partial t = \nu\nabla^2\mathbf{B}$, leads to a decay-time λ^2/ν, where λ is a length characteristic of the dimensions of the conductor. This may be regarded as the time necessary for the field to diffuse through a distance λ. One may say that the field diffuses with a speed ν/λ. The equation (3) is interpreted to mean that the decay-time of the $\hat{\mathbf{B}}$-field <u>for an observer moving with the fluid</u> is λ^2/ν, and the speed of diffusion relative to this observer is of the order

$$\frac{\nu}{\lambda} = \frac{v}{R_m} \qquad , \tag{30}$$

where R_m is the magnetic Reynolds number (Elsasser, 1954). During the time a fluid particle has moved a distance comparable to the scale of the velocity field, λ, the B-field has diffused through a distance

$$\ell = \lambda/R_m \ . \tag{31}$$

We see that

$$\frac{\partial\hat{B}_k}{\partial x_j} \lesssim \frac{\hat{B}}{\ell} \quad \frac{R_m\hat{B}}{\lambda} \tag{32}$$

$$\frac{\partial^2 B_k}{\partial x_j \partial x_j} \sim \frac{\hat{B}}{\ell^2} \quad \frac{R_m^2\hat{B}}{\lambda^2}$$

whence also,

$$A = |\mathbf{A}| \sim \frac{R_m\hat{B}\,\delta}{\lambda^2}$$

$$\left|(\nabla^2\hat{\mathbf{B}}\cdot\nabla^\circ)\,\mathbf{r}\right| \sim \frac{R_m^2\hat{B}\,\delta}{\lambda^2} \tag{33}$$

A comparison of terms of the right of (9') gives

$$\frac{C}{\left|(\nabla^2\hat{\mathbf{B}}\cdot\nabla^\circ)\,\mathbf{r}\right|} \sim \frac{1}{R_m^2} \tag{34}$$

$$\frac{A}{\left|(\nabla^2\hat{\mathbf{B}}\cdot\nabla^\circ)\,\mathbf{r}\right|} \sim \frac{1}{R_m} \ .$$

Hence, if the flow is reasonably smooth both $\mathcal{A}$ and $\mathcal{C}$ may be neglected in comparison with the first term in (9′) provided

$$R_m \gtrsim 10 \; . \qquad (35)$$

The divergence condition, (14), is satisfied to the same order of small quantities, as discussed in Appendix IV B.

IV. The Substantial Diffusion Equation

a. Preliminaries; nature of the problem

Suppose there is no interaction between the components of $\mathbf{B}$; this means that $(\mathbf{B}\cdot\nabla)\mathbf{v} = 0$, and the induction equation (3-1) is reduced to

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \nu\nabla^2\mathbf{B}$$
$$\text{or } \frac{d\mathbf{B}}{dt} = \nu\nabla^2\mathbf{B}. \tag{1}$$

This is precisely the substantial diffusion equation encountered in sec. 3. It arises, in three cases: it is the induction equation when

$$(\mathbf{B}\cdot\nabla)\mathbf{v} = 0 ; \tag{2}$$

it is approximately the induction equation for large-scale velocity fields, for when $\mathbf{v}$ varies slowly, that is when

$$(\mathbf{B}\cdot\nabla)\mathbf{v} \ll (\mathbf{v}\cdot\nabla)\mathbf{B} \tag{3}$$

so that it may be dropped from equation (1-8); it is the equation for $\hat{\mathbf{B}}$ in the partial integral of the complete induction equation when neither (2) nor (3) is true.

We propose in this section to obtain solutions of (1). The symbol $\mathbf{B}$ will be used throughout, with the understanding it may be replaced by $\hat{\mathbf{B}}$ when appropriate. Whenever the solutions obtained are intended to represent the complete magnetic field and not just $\hat{\mathbf{B}}$, they are subject to either (2) or (3), and this must be verified for particular solutions. The field is continually spreading both by diffusion and by its being (partially) dragged with the fluid. It is not difficult, however, to construct special $\mathbf{v}$ and $\mathbf{B}$ configurations which because of their geometries satisfy (2) at all times. If $(\mathbf{B}\cdot\nabla)\mathbf{v}$ is merely small, initially, then (3) cannot hold indefinitely; nevertheless, it should remain true for times of the order of the period of fluid motion, v/λ. This is because in the range of R_m where both diffusion and induction are important, the spreading effects of diffusion and "differential dragging" tend to oppose one another. (This assertion is based on the familiar "entropy" argument: that the general effect of diffusion is to smooth out inhomogeneities in the field---in this case inhomogeneities caused by the fluid motion. See also Inglis (1955) who cites an example to show the extent to which magnetic field lines can be stretched by "dragging" before their "slipping" velocity relative to the fluid equals the fluid velocity. Lundquist (1952) also gives an example.

It has already been remarked that the substantial diffusion equation may be regarded as a first-order wave equation with dissipation. A closer examination, however, reveals that (1) is an equation of the elliptic rather than hyperbolic type (Sommerfeld, 1949, p. 38), and mathematically it is more akin to a diffusion equation than a wave equation. It is in this direction, therefore, that we look for a solution. Physically the equation represents a combination of propogation and diffusion in an inhomogeneous anisotropic "medium"; consequently, it will not be surprising to find its solution cumbersome.

b. The substantial diffusion equation as an inhomogeneous diffusion equation

Trying to solve the equation, even for simple fluid flows, dispels any notion of there existing bounded solutions with variables separated. A Fourier analysis and an application of the WKB approximation for slowly varying $\mathbf{V}$ were also unsuccessful in obtaining physically meaningful solutions.

We therefore choose to look upon (1) as an inhomogeneous diffusion equation,

$$\nabla^2 \mathbf{B} - \frac{1}{\nu}\frac{\partial \mathbf{B}}{\partial t} = \left(\frac{\mathbf{V}}{\nu}\cdot\nabla\right)\mathbf{B} . \qquad (1^1)$$

From this point of view, $\mathbf{B}$ has "sources" $-\nu^{-1}(\mathbf{V}\cdot\nabla)\mathbf{B}$ which arise from its interaction with the fluid.

Using Green's function for the diffusion equation one can show that

$$\mathbf{B}(\mathbf{r},t) = -\nu^{-1}\int_0^{t^+} dt \int_V d^3\mathbf{r}'\, G(\mathbf{v}'\cdot\nabla')\,\mathbf{B}' \tag{4}$$
$$+\nu^{-1}\int_V d^3\mathbf{r}'\,(G\,\mathbf{B}')_{t'=0} + \int_0^{t^+} dt' \int_S \left(G\frac{\partial \mathbf{B}'}{\partial n'} - \mathbf{B}'\frac{\partial G}{\partial n'}\right) ds',$$

in which $\mathbf{B}$ is the field within a volume V bounded by a surface S; $G(\mathbf{r},t|\mathbf{r}',t)$ is the appropriate Green's function, $\mathbf{v}'$ and $\mathbf{B}'$ denote $\mathbf{v}(\mathbf{r}',t')$ and $\mathbf{B}(\mathbf{r}',t')$ respectively, and t+ means $t+\varepsilon^2$, $\varepsilon\to 0$. The details of this operation will be found in Appendix II.

Assuming that $G \sim 1/r$ and $B \sim 1/r$, at least, as the surface integral vanishes as S receeds to infinity, and in the infinite domain

$$\mathbf{B}(\mathbf{r},t) = -\nu^{-1}\int_0^{t^+} dt \int d^3\mathbf{r}'\; G(\mathbf{v}'\cdot\nabla')\,\mathbf{B}' \tag{5}$$
$$+\nu^{-1}\int d^3\mathbf{r}'(G\,\mathbf{B}')_{t=0}\,.$$

It follows that

$$(\mathbf{v}\cdot\nabla)\mathbf{B} = -\nu^{-1}\int_0^{t^+} dt' \int d^3\mathbf{r}'(\mathbf{v}\cdot\nabla G)\,(\mathbf{v}'\cdot\nabla')\,\mathbf{B}' \tag{6}$$
$$+\nu^{-1}\int d^3\mathbf{r}'\left[(\mathbf{v}\cdot\nabla G)\mathbf{B}'\right]_{t'=0}\,.$$

The last integral is a known function or rather functional, in terms of $\mathbf{B}(\mathbf{r},0)$, which will be designated by

$$\Gamma(\mathbf{r},t) = \nu^{-1}\int d^3\mathbf{r}'\left[\mathbf{v}\cdot\nabla G(\mathbf{r},t|\mathbf{r}',0)\right]\mathbf{B}(\mathbf{r}',0). \tag{7}$$

Then (6) has the form of a Volterra-Fredholm equation for $(\mathbf{v}\cdot\nabla)\mathbf{B}$, viz.

$$(\mathbf{v}\cdot\nabla)\mathbf{B}=\Gamma(\mathbf{r},t)+\int_0^{t^+}dt'\int d^3\mathbf{r}'\,K(\mathbf{r},t\,|\,\mathbf{r}',t')(\mathbf{v}'\cdot\nabla')\mathbf{B}', \tag{8}$$

whose kernel is

$$K(\mathbf{r},t\,\mathbf{r}',t)=-\nu^{-1}\mathbf{v}(\mathbf{r})\cdot\nabla G(\mathbf{r},t\,|\,\mathbf{r}',t'). \tag{9}$$

At this point it becomes necessary to state something more about the Green's function. The appropriate Green's function for the infinite domain can be shown to be the impulse function

$$G(R\,|\,\tau)=\nu U(\tau)\,(4\pi\nu\tau)^{-n/2}\exp(-R^2/4\nu\tau) \tag{10}$$

$$R=|\mathbf{r}-\mathbf{r}'|,\quad \tau=t-t\ ,$$

where U is the unit step function

$$U(\tau)=\begin{cases}0, & \tau<0\\ 1, & \tau>0.\end{cases}$$

(Morse and Feshbach, 1953, p. 894) Certainly, then, our assumption about the behavior of G at ∞ was justified. Furthermore, since G=0 for all $t'>t$, the time integration in (8) can equally well be extended over all t, so we may write

$$(\mathbf{v}\cdot\nabla)\mathbf{B}=\Gamma(\mathbf{r},t)+\int_0^{\infty}dt'\int d^3\mathbf{r}'\,K(\mathbf{r},t\,|\,\mathbf{r}',t')(\mathbf{v}'\cdot\nabla')\mathbf{B}', \tag{11}$$

which is an inhomogeneous Fredholm equation of the second kind. The kernel is asymmetric:

$$K=\frac{U(\tau)\,\mathbf{v}\cdot\mathbf{R}\,\exp(-R^2/4\nu\tau)}{2\nu\tau\,(4\pi\nu\tau)^{n/2}} \tag{12}$$

$$\mathbf{R}=\mathbf{r}-\mathbf{r}'.$$

Fredholm equations for a function of more than one variable are seldom discussed in the textbooks; however, they can be solved using straightforward generalizations of the standard methods used on equations for a function of one variable (Courant and Hilbert, 1953, page 152). One method of solution is found in Appendix IV. Finally, let

$$\mathbf{\Phi}(\mathbf{r},t) = (\mathbf{v}\cdot\nabla)\mathbf{B} \tag{13}$$

be the solution so obtained; then substitution into (5) gives

$$\mathbf{B}(\mathbf{r},t) = -\nu^{-1}\int_0^\infty dt' \int d^3\mathbf{r}' G(\mathbf{r},t|\mathbf{r}',t')\,\mathbf{\Phi}(\mathbf{r}',t') \tag{14}$$

$$+\nu^{-1}\int d^3\mathbf{r}' G(\mathbf{r},t/\mathbf{r}',0)\,\mathbf{B}(\mathbf{r}',0)\ .$$

It is proved in Appendix IVA that

$$\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0 \text{ provided } \nabla\cdot\mathbf{B}(\mathbf{r},0) = 0\ . \tag{15}$$

c. The substantial diffusion equation reduced to an inhomogeneous Helmholtz equation

Examination shows that there exist no physically meaningful solutions of (1) with variables separated,

$$B_\alpha(\mathbf{r},t) = X_\alpha(x_1)Y_\alpha(x_2)Z_\alpha(x_3)T(t)$$

(no summation on greek indices). This, however, does not preclude the existence of solutions composed of a sum of terms, each term of which has variables separated. In an effort, then, to find solutions of (1) less awkward than (14) we look for solutions of the form

$$\mathbf{B}(\mathbf{r},t) = \int dk\, c(k)T(t;k)\mathbf{F}(\mathbf{r};k), \tag{16}$$

a superposition of terms with variables separated.
Then

$$\nu \nabla^2 \mathbf{B} - \frac{\partial \mathbf{B}}{\partial t} - (\mathbf{V}\cdot\nabla)\mathbf{B}$$
$$= \int dk\, c(k) \left[T(\nu \nabla^2 \mathbf{F} - (\mathbf{V}\cdot\nabla)\mathbf{F} - \mathbf{F}\frac{dT}{dt} \right], \tag{17}$$

which vanishes if

$$\begin{cases} \frac{dT}{dt} + \nu k^2 T = 0 \\ \nabla^2 \mathbf{F} + k^2 \mathbf{F} = \nu^{-1}(\mathbf{V}\cdot\nabla)\mathbf{F}\, . \end{cases} \tag{18}$$

(18a) gives $\qquad T = e^{-\nu k^2 t}$, $\qquad$ (19)
as usual.

The plan now is to look upon (18b) as an inhomogeneous Helmholtz equation. Doing this one may show that

$$\mathbf{F}(\mathbf{r};k) = -\nu^{-1}\int_V d^3\mathbf{r}'\, G(\mathbf{V}'\cdot\nabla')\mathbf{F}'$$
$$+ \int_S \left(G\frac{\partial \mathbf{F}'}{\partial n'} - \mathbf{F}'\frac{\partial G}{\partial n'}\right) ds'\, . \tag{20}$$

The details of this operation are contained in Appendix III. G is the Green's function for the Helmholtz equation and may be taken as

$$G(\mathbf{r}|\mathbf{r}';k) = R^{-1}\exp(ikR) \tag{21}$$

(Morse and Feshbach, 1953, p. 891). As S recedes to infinity the surface integral vanishes if F goes at least as r^{-1} as $r \to \infty$. Assume this is so. In the infinite domain, then,

$$\mathbf{F}(\mathbf{r};k) = -\nu^{-1}\int d^3\mathbf{r}'\, G(\mathbf{r}|\mathbf{r}';k)(\mathbf{V}'\cdot\nabla')\mathbf{F}' \, . \tag{22}$$

This is a particular solution of (18b); to it we may add any solution, $\check{\mathbf{F}}$, of the homogenous Helmholtz equation,

$$\nabla^2 \check{\mathbf{F}} + k^2 \check{\mathbf{F}} = 0 \quad . \tag{23}$$

Doing this will guarantee our obtaining a solution $\mathbf{F}$ with zero divergence (a subject discussed in Appendix IVC). We have now

$$\mathbf{F}(\mathbf{r};k) = \check{\mathbf{F}}(\quad ;k) - \nu^{-1} \int d^3\mathbf{r}' G(\mathbf{r}|\mathbf{r}';k)(\mathbf{v}' \cdot \nabla')\mathbf{F}' \quad . \tag{24}$$

It follows at once that

$$(\mathbf{v} \cdot \nabla)\mathbf{F} = (\mathbf{v} \cdot \nabla)\check{\mathbf{F}} - \nu^{-1} \int d^3\mathbf{r}' (\mathbf{v} \cdot \nabla G)(\mathbf{v}' \cdot \nabla')\mathbf{F}', \tag{25}$$

which is an inhomogeneous Fredholm equation of the second kind for $(\mathbf{v} \cdot \nabla)\mathbf{F}$, with the asymmetric kernel $-\nu^{-1}\mathbf{v} \cdot \nabla G$. It can be solved using extensions of standard methods. Let

$$\mathbf{\Lambda}(\mathbf{r};\ k) = (\mathbf{v} \cdot \nabla)\mathbf{F} \tag{26}$$

be the solution so obtained; putting this into (24) gives

$$\mathbf{F}(\mathbf{r};k) = \check{\mathbf{F}}(\mathbf{r};k) - \nu^{-1} \int d^3\mathbf{r}' G(\mathbf{r}|\mathbf{r}';k)\, \mathbf{\Lambda}(\mathbf{r}';k) \quad . \tag{27}$$

The general (though not necessarily complete) solution of (1) is

$$\mathbf{B}(\mathbf{r},t) = \int dk\, c(k)\, \mathbf{F}(\mathbf{r};k)\, \exp(-\nu k^2 t). \tag{28}$$

Furthermore,

$$\mathbf{B}(\mathbf{r},0) = \int dk\, c(k)\, \mathbf{F}(\mathbf{r};k), \tag{29}$$

which is a Fredholm equation of the first kind, with the kernel $\mathbf{F}(\mathbf{r};k)$. Its solution (if one exists), obtained by standard methods, gives the spectrum function $c(k)$ in terms of the initial field $\mathbf{B}(\mathbf{r},0)$. As to the divergence of $\mathbf{B}$, note that

$$\nabla \cdot \mathbf{B} = \int dk\, c(k)(\nabla \cdot \mathbf{F})\exp(-\nu k^2 t),$$

which vanishes identically only if $\nabla \cdot \mathbf{F} = 0$. In Appendix IVC it is demonstrated that

$$\nabla \cdot \mathbf{B} = 0 \quad \text{provided} \quad \nabla \cdot \check{\mathbf{F}} = 0 \quad . \tag{30}$$

The order of the definite integrals involved has been reduced by one by using this quasi-separation of variables in (16). It has been done, however, at the expense of assuming that there exist solutions of (1) of the form (16) which are regular at ∞ and which allow the spectrum function to be determined through (29). Whether or not this is so must evidently be decided in individual cases. In obtaining solution (14) its form was in no wise restricted; hence there can be no doubt that it vanishes at least as r^{-2} at infinity, whenever the initial field is confined to a finite region(because from a distance, then, the field appears to arise from multipole sources).

It seems now natural to inquire if (1) could not be treated as an inhomogeneous wave equation to which the familiar Kirchhoff method of integration might be applied. This method is not applicable, it turns out, because of the intrinsically anisotropic character of the propagation which (1) describes.

d. The solutions in Lagrangian form

Consider the subject of section 4a. Once the solution $\boldsymbol{\Phi}$ of (11) is known one may proceed directly to find $\mathbf{B}$ in Lagrangian form. Using (2-3), (2-5), and (13) we may write

$$\left(\frac{d\mathbf{r}}{dt}\cdot\nabla\right)\mathbf{B} = \boldsymbol{\Phi}, \tag{31}$$

where the use of the total derivative means we are thinking of $\boldsymbol{\Phi}$ as expressed in terms of Lagrangian variables. Then, since $d\varphi = d\,\mathbf{r}\cdot\nabla\varphi$ is the change in φ in the direction of $d\mathbf{r}$,

$$d\mathbf{B} = (d\,\mathbf{r}\cdot\nabla)\mathbf{B} = \boldsymbol{\Phi}\left[\mathbf{r}(\mathbf{r}^{o},t),t\right]dt \tag{32}$$

gives the change in $\mathbf{B}$ arising from a displacement $d\mathbf{r}$ along a Lagrangian trajectory (and includes the effects of both time- and space-changes). As a result

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r}^{o},0) + \int_{0}^{t}\boldsymbol{\Phi}(\mathbf{r}^{o},t)dt\ . \tag{33}$$

The Lagrangian and Eulerian formulations are merely different ways of expressing the same thing, and because of this the conclusion (15) about div$\mathbf{B}$ is still valid, and

$$\nabla \cdot \mathbf{B} = 0 \text{ if } \nabla^{o} \cdot \mathbf{B}(\mathbf{r}^{o},0) = 0 \tag{34}$$

Although it is usually more convenient to have solutions expressed in Eulerian terms, it may be desirable to settle for a Lagrangian form and thus avoid the complicated multiple integrations involved in (14).

The solution which is the subject of section 4c may also be expressed in Lagrangian form, at a considerable saving of labor. From (26) we have

$$d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\mathbf{F} = \Lambda\left[\mathbf{r}(\mathbf{r}^{o},t)\right] dt, \tag{35}$$

and

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(\mathbf{r}^{o}) + \int_{0}^{t} \Lambda(\mathbf{r}^{o},t)\,dt\,. \tag{36}$$

Besides requiring that $\nabla \cdot \mathbf{F} = 0$, as in (30), it is necessary to have

$$\nabla^{o} \cdot \mathbf{F}(\mathbf{r}^{o}) = 0, \tag{37}$$

because the term $\mathbf{F}(\mathbf{r}^{o})$ is arbitrary as far as the Lagrangian integration is concerned.

e. The pure-diffusion solutions

We close this discussion with an account of one class of solutions of (1) available at once. This class is obtained by setting

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nu \nabla^{2} \mathbf{B} \\ (\mathbf{v} \cdot \nabla)\mathbf{B} &= 0\,. \end{aligned} \tag{38}$$

$\mathbf{B}$ obeys the conventional diffusion equation and in addition, according to (38b), must not vary in the direction of $\mathbf{v}$. From (38) we see that $\nabla \cdot \mathbf{B}$ likewise obeys a diffusion equation so that

$$\nabla \cdot \mathcal{B} = 0 \quad \text{if} \ (\nabla \cdot \mathcal{B})_{t=0} = 0. \tag{29}$$

Let

$$U_1(\mathbf{r}) = k_1, \ U_2(\mathbf{r}) = k_2 \tag{40}$$

be linearly independent ($\nabla U_1 \times \nabla U_2 \neq 0$) and integrals of

$$d\mathbf{r} \times \mathbf{V} = 0 \quad , \tag{41}$$

This means the intersection of the surfaces defined by (40) falls along the streamlines; and if $U_3 = K \nabla U_1 \times \nabla U_2$, we have (U_1, U_2, U_3) forming a system of curvilinear coordinates. As a consequence (38b) is satisfied if

$$\mathcal{B} = \mathcal{B}(U_1, U_2, t) = e^{-\nu^2 t} \mathcal{F}(U_1, U_2). \tag{42}$$

The solution of (38a) is now a question of separability. The space part of the solution satisfied the <u>vector</u> Helmholtz equation, which for other than cartesian coordinates should be written

$$\nabla \times \nabla \times \mathcal{F} = k^2 \mathcal{F}, \quad \nabla \cdot \mathcal{F} = 0. \tag{43}$$

Its separability is essentially limited to six different coordinate systems (Morse and Feshbach, 1953, p. 1767); viz. circular, elliptic, and parabolic cylinder coordinates; spherical and conical coordinates; and, of course, rectangular coordinates. Thus equation (38) can be solved simultaneously if the streamlines of $\mathbf{V}$ coincide with one set of coordinate lines for any of these six systems. If they coincide, say, with the U_3-lines (the lines on which U_1 and U_2 are constant) then $\mathcal{F} = \mathcal{F}(U_1, U_2)$ may be any divergence-free solution of the two dimensional Helmholtz equation in U_1 and U_2. The solution of (43) is completely treated elsewhere (Morse and Feshbach, Vol, II, pp. 1762-1767). The simplest example that comes to mind is where

$$\mathbf{V} = \mathbf{e}_1 v \ \text{and} \ \mathcal{B} = e^{-\nu^2 t} \cdot \mathcal{F}(x_2, x_3)$$

with $\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + k^2\right)\mathbf{F} = 0, \quad \nabla \cdot \mathbf{F} = 0.$

These pure-diffusion and non-wave-like solutions are of limited physical interest. To require in addition that $(\mathbf{B} \cdot \nabla)\mathbf{v} = 0$ would be altogether too restrictive; hence they are of interest not as integrals of the induction equation, but as possible $\hat{\mathbf{B}}$-fields, with $\mathbf{B}$ given by (3-2).

Appendix I Results of General Interest for a Lagrangian Description of Fluid Flow.

Let $\mathbf{r}^o$ locate the initial position of a point moving with the fluid. Its position after a time t is given by the trajectory $\mathbf{r} = \mathbf{r}(\mathbf{r}^o, t)$, where $\mathbf{r}$ is a solution of

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \tag{1}$$

$\mathbf{v}$ being the fluid velocity. In the Lagrangian formulation $\mathbf{r}^o$ and t are the independent variables and $\partial/\partial t$ and d/dt commute with the $\partial/\partial x_i^o$; however, the $\partial/\partial x_i^o$ and $\partial/\partial x_j$ do not commute. If operating on a function expressed in Eulerian terms, then $d\mathbf{F}/dt$ means $\partial\mathbf{F}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{F}$, the familiar substantial derivative; but if operating on a function expressed in Lagrangian terms, $d\mathbf{F}/dt$ means simply $\partial\mathbf{F}/\partial t$.

The following results hold.

$$J\frac{\partial\mathbf{r}^o}{\partial x_\alpha} = (\nabla^o x_\beta) \times (\nabla^o x_\gamma) = \nabla^o \times (x_\beta \nabla^o x_\gamma) \tag{2}$$

where

$$J = \left| \frac{\partial x_i}{\partial x_j^o} \right| \tag{3}$$

is the Jacobian or functional determinant of the transformation $\mathbf{r}^o \rightarrow \mathbf{r}(\mathbf{r}^o, t)$, and (α, β, γ) is a cyclic permutation on (1,2,3). If

$$J^{-1} = \left| \frac{\partial x_i^o}{\partial x_j} \right| \tag{4}$$

denotes the Jacobian of the inverse transformation $\mathbf{r} \rightarrow \mathbf{r}^o(\mathbf{r}, t)$, then similarly

$$J^{-1}\frac{\partial\mathbf{r}}{\partial x_\alpha^o} = (\nabla x_\beta^o) \times (\nabla x_\gamma^o) = \nabla \times (x_\beta^o \nabla x_\gamma^o). \tag{5}$$

When the flow is incompressible

$$\nabla\cdot\mathbf{v} = 0 \tag{6}$$

and $$J = J^{-1} = 1, \tag{7}$$

which means that
$$\nabla \cdot \frac{\partial \mathbf{r}}{\partial x_i^o} = \nabla^o \cdot \frac{\partial \mathbf{r}^o}{\partial x_j} = 0. \tag{8}$$

(8) is the Lagrangian equivalent of the Eulerian statement (6) expressing incompressibility.

Equation (1) can be used in special cases to derive conservation theorems for $\mathbf{r}$. Suppose $\mathcal{L}^o$ denotes any linear differential operator with terms in the $\partial/\partial x_i^o$. Then, using our commutation rules

$$\frac{d}{dt} \mathcal{L}^o \mathbf{r} = \mathcal{L}^o \mathbf{v},$$

and if
$$\mathcal{L}^o \mathbf{v} = 0$$

we would have
$$\frac{d}{dt} \mathcal{L}^o \mathbf{r} = \frac{\partial}{\partial t} \mathcal{L}^o \mathbf{r} = 0,$$

whence $\mathcal{L}^o \mathbf{r} = \text{const.} = \mathcal{L}^o \mathbf{r}^o$ for any fluid particle. By way of example, if in particular cases $\nabla^o \cdot \mathbf{v} = 0$, it follows that

$$\nabla^o \cdot \mathbf{r} = \nabla^o \cdot \mathbf{r}^o = 3.$$

For such purposes it may be convenient to express $\nabla^o \cdot \mathbf{v}$ in any of the equivalent forms

$$\frac{\partial v_i}{\partial x_i^o} = \frac{\partial x_j}{\partial x_i^o} \frac{\partial v_i}{\partial x_j} = \frac{\partial \mathbf{r}}{\partial x_i^o} \cdot \nabla v_i = \nabla v_1 \cdot (\nabla x_2^o) \times (\nabla x_3^o) + \ldots,$$

etc.

Appendix II. Integration of the inhomogeneous diffusion equation

The substantial diffusion equation of sec. 4 is to be written as
$$\nabla^2 \mathbf{B} - \nu^{-1} \frac{\partial \mathbf{B}}{\partial t} = \nu^{-1} (\mathbf{v} \cdot \nabla) \mathbf{B} \tag{1}$$

and regarded as an inhomogeneous diffusion equation. Let

(1) be multiplied by a function G which is essentially an integrating factor and remains to be determined. Then let the quantities $\mathcal{B}\nabla^2 G$ and $\nu^{-1}\mathcal{B}\,\partial G/\partial t$ be both added and subtracted on the left. After rearranging terms and integrating there obtains

$$\int_0^{t^+} dt' \int_V d^3 r' \left[G\, \nabla'^2 \mathcal{B}' - \mathcal{B}' \nabla'^2 G\right] + \int_0^{t^+} dt' \int_V d^3 r' \left[\mathcal{B}\, \nabla'^2 G + \frac{\mathcal{B}'}{\nu} \frac{\partial G}{\partial t'}\right]$$

$$-\nu^{-1} \int_0^{t^+} dt' \int_V d^3 r' \left[\mathcal{B}' \frac{\partial G}{\partial t'} + G\, \frac{\partial \mathcal{B}'}{\partial t'}\right] = \nu^{-1} \int_0^{t^+} dt' \int_V d^3 r'\, G(\mathbf{v}' \cdot \nabla')\, \mathcal{B}' . \tag{2}$$

$\mathcal{B}(r',t')$ and $\mathbf{v}(r',t')$ are represented by $\mathcal{B}'$ and $\mathbf{v}'$, and t^+ means $t + \varepsilon^2$, $\varepsilon \to 0$; the volume V remains as yet arbitrary.

Now, let $G(r,t\,|\,r',t')$ be the response of the system at (r,t) to an instantaneous point source of unit strength located at (r',t'), i.e.,

$$\nabla^2 G - \nu^{-1} \frac{\partial G}{\partial t} = -\,\delta(r - r')\, \delta(t-t\ ); \tag{3}$$

and require that

$$G(r,t\,|\,r',t') = 0, \quad t < t' \tag{4}$$

(this is a causality statement, required by the unidirectionality in time implicit in the diffusion equation). It is a consequence of these two conditions that

$$G(r,t\,|\,r',t') = G(r',-t'\,|\,r,-t), \tag{5}$$

(See e.g. Morse and Feshbach, Vol. I., p. 858) which is a statement of reciprocity and is rather obvious from physical considerations. It follows that

$$\nabla^2 G + \nu^{-1} \frac{\partial G}{\partial t} = -\,\delta(r - r')\, \delta(t-t\ ). \tag{6}$$

Returning to (2) and using this result gives

$$\int_0^t dt' \int_V d^3\boldsymbol{r}' \left[\nabla'^2 G + \nu^{-1}\frac{\partial G}{\partial t'}\right] \mathcal{B}(\boldsymbol{r}',t') = -\mathcal{B}(\boldsymbol{r},t). \tag{7}$$

As to the last integral on the left of (2), reverse the order of integration and obtain

$$-\nu^{-1}\int_V d^3\boldsymbol{r}' \int_0^{t^+} dt' \frac{\partial}{\partial t'}(G\mathcal{B}') = -\nu^{-1}\int_V d^3\boldsymbol{r}' \Big[G(\boldsymbol{r},t\,|\,\boldsymbol{r}',t')\,\mathcal{B}(\boldsymbol{r}',t')\Big]_0^{t^+} \tag{8}$$

But according to (4)

$$G(\boldsymbol{r},t\,|\,\boldsymbol{r}',t^+) = 0 \tag{9}$$

(which was the reason for integrating to t^+); hence,

$$(8) = \nu^{-1}\int d^3\boldsymbol{r}'(G\mathcal{B}')_{t'=0} \,. \tag{10}$$

Finally, we apply the symmetrical form of Green's theorem to the first integral in (2):

$$\int_0^{t^+} dt' \int_V d^3\boldsymbol{r}' \left[G\,\nabla'^2\mathcal{B}' - \mathcal{B}'\,\nabla'^2 G\right] = \int_0^{t^+} dt' \int_S \left[G\,\frac{\partial\mathcal{B}'}{\partial n'} - \mathcal{B}'\frac{\partial G}{\partial n'}\right] ds' \,, \tag{11}$$

after which (2) becomes

$$\mathcal{B}(\boldsymbol{r},t) = -\nu^{-1}\int_0^{t^+} dt' \int_V d^3\boldsymbol{r}'\, G(\boldsymbol{v}'\cdot\nabla')\mathcal{B}' + \nu^{-1}\int_V d^3\boldsymbol{r}'(G\mathcal{B}')_{t=0} + \int_0^{t^+} dt' \int_S \left[G\,\frac{\partial\mathcal{B}'}{\partial n'} - \mathcal{B}'\frac{\partial G}{\partial n'}\right] ds' \,, \tag{12}$$

which is equation (4-4).

(12) has a simple physical interpretation. The

contribution to $\mathcal{B}$ from "sources" exterior to the region enclosed by $\mathcal{S}$ is expressed by the surface integral. The contribution from within V comes from an integration over a continuous distribution of sources having strengths $-(\boldsymbol{v}\cdot\nabla)\mathcal{B}$ and $\mathcal{B}(\boldsymbol{r},0)$; the latter are instantaneous sources.

The essential steps in this integration may be found in Morse and Feshbach(1953, pp. 859 ff.) or (1955, Chap. VII).

Appendix III. Integration of the Inhomogeneous Helmholtz Equation.

Equation (4-18b) reads

$$\nabla^2 \mathbf{F} + k^2 \mathbf{F} = \nu^{-1}(\boldsymbol{v}\cdot\nabla)\mathbf{F}, \tag{1}$$

a Helmholtz equation with "sources" $-\nu^{-1}(\boldsymbol{v}\cdot\nabla)\mathbf{F}$. Let G be the response to a unit point source at $\boldsymbol{r}'$, then

$$\nabla^2 G + k^2 G = -\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2}$$

G may be used as an integrating factor. Multiply (1) by G and (2) by $\mathbf{F}$, subtract the two equations and integrate; one obtains

$$\int_V d^3\boldsymbol{r}'(\mathbf{F}'\nabla'^2 G - G\nabla'^2\mathbf{F}') = -\int_V d^3\boldsymbol{r}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\mathbf{F}(\boldsymbol{r}') - \nu^{-1}\int_V d^3\boldsymbol{r}' G(\boldsymbol{v}'\cdot\nabla')\mathbf{F}'. \tag{3}$$

On using Green's theorem in its symmetrical form,

$$\mathbf{F}(\boldsymbol{r}) = -\nu^{-1}\int_V d^3\boldsymbol{r}' G(\boldsymbol{v}'\cdot\nabla')\mathbf{F}' + \int_S \left(\mathbf{F}'\frac{\partial G}{\partial n'} - G\frac{\partial \mathbf{F}'}{\partial n'}\right)ds . \tag{4}$$

This is equation (4-20).

Appendix IV. On the Divergence of Solutions of the Substantial Diffusion Equation.

A. The substantial diffusion equation, repeated from sec. 4 is

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{B} = \nu \nabla^2 \mathbf{B}, \tag{1}$$

in which the symbol $\mathbf{B}$ may represent $\hat{\mathbf{B}}$ also, when appropriate. From (2-2) and (2-12) we see that

$$\nabla \cdot (\mathbf{V} \cdot \nabla)\mathbf{B} = \mathbf{V} \cdot \nabla(\nabla \cdot \mathbf{B}) + \nabla \cdot (\mathbf{B} \cdot \nabla)\mathbf{V} \tag{2}$$

Assume for the moment that

$$(\mathbf{B} \cdot \nabla)\mathbf{V} = 0 \quad \text{or is negligible,} \tag{3}$$

which must be the case if $\mathbf{B}$ is an integral of the induction equation, according to section 4a. On taking the divergence of (1),

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) + \mathbf{V} \cdot \nabla(\nabla \cdot \mathbf{B}) = \nu \nabla^2 (\nabla \cdot \mathbf{B}), \tag{4}$$

so that $\mathbf{B}$ and its divergence satisfy the same equation.

Let for a moment the symbol $\mathbf{b}$ represent both $\mathbf{B}$ and div $\mathbf{B}$. We then write

$$\nabla^2 \mathbf{b} - \nu^{-1} \frac{\partial \mathbf{b}}{\partial t} = \nu^{-1} (\mathbf{V} \cdot \nabla)\mathbf{b} \tag{5}$$

and discuss both relations simultaneously.

Suppose the "source" function on the right is initially zero, then momentarily (5) reduces to the homogeneous diffusion equation,

$$\frac{\partial \mathbf{b}}{\partial t} = \nu \nabla^2 \mathbf{b},$$

which has solutions of the form

$$\mathbf{b} = e^{-\nu t}\, \mathbf{b}_{t=0} \,.$$

If now $\mathbf{b}_{t=0}=0$, this suggests that the sources remain zero. This must certainly be true for physically meaningful solutions, $\mathbf{B}$, for the magnetic field; if no magnetic field is present originally, none can be generated spontaneously by the fluid motion.

To substantiate analytically this physical interpretation of (5) one must inquire into the nature of the solutions of the integral equation to which (5) leads. From (4-14) we have

$$\mathbf{b}(\mathbf{r},t) = -\nu^{-1}\int d\rho' G(\mathbf{r},t\,|\,\mathbf{r}',t')\Pi(\mathbf{r}',t') + \nu^{-1}\int d^3\mathbf{r}' G(\mathbf{r},t\,|\,\mathbf{r}',0)\,\mathbf{b}(\mathbf{r}',0), \tag{6}$$

in which $d\rho' = dt'd^3\mathbf{r}'$, G is the appropriate Green's function, and Π is by (4-11) a solution of

$$\Pi(\mathbf{r},t) = \Lambda(\mathbf{r},t) + \int d\rho' K(\mathbf{r},t\,|\,\mathbf{r}',t')\Pi(\mathbf{r}',t'), \tag{7}$$

where

$$\begin{cases} \Lambda(\mathbf{r},t) = \int d^3\mathbf{r}' K(\mathbf{r},t\,|\,\mathbf{r}',0)\,\mathbf{b}(\mathbf{r}',0) \\ K(\mathbf{r},t\,|\,\mathbf{r}',t') = -\nu^{-1}\mathbf{v}\cdot\nabla G(\mathbf{r},t\,|\,\mathbf{r}',t') \,. \end{cases} \tag{8}$$

is to be regarded as a known function. In what follows $\Pi' = \Pi(\mathbf{r}',t')$ etc., and ρ will denote the combination of variables $(\mathbf{r},t)$.

One approach to solving (7) is the method of iterations (Page, sec. 9.2; Courant - Hilbert, Ch. 3. sec. 6). Choose any finite continuous vector function as the zero order approximation for Π; let it be called Π^0. Then the first-order approximation for Π is

$$\Pi^1(\rho) = \Lambda(\rho) + \int d\rho' K(\rho|\rho')\Pi^0(\rho'); \tag{9}$$

$$\Pi^2(\rho) = \Lambda(\rho) + \int d\rho'' K(\rho|\rho'')\,\Lambda(\rho'') + \int d\rho' K_2(\rho|\rho')\,\Pi^0(\rho') \tag{10}$$

where $$K_2(\rho|\rho') = \int d\rho'' K(\rho|\rho'') K(\rho''|\rho) \, . \tag{11}$$

If this iteration process is continued indefinitely and we define the iterated kernels

$$K_n(\rho|\rho') = \int d\rho'' K(\rho|\rho'') K_{n-1}(\rho''|\rho), \quad n \geqslant 2$$
$$K_1(\rho|\rho') = K(\rho|\rho'), \tag{12}$$

there obtains for the $n^{\underline{th}}$ approximation

$$\Pi^n(\rho) = \Lambda(\rho) + \sum_{i=1}^{n-1} \int d\rho' K_i(\rho|\rho') \Lambda(\rho') + \int d\rho' K_n(\rho|\rho') \Pi^0(\rho') . \tag{13}$$

It is shown in treatises on integral equation (Whittaker and Watson, 1920, p. 222) that regardless of the choice of a trial solutic Π^0, the last term in (13) approaches zero and the series converges uniformly. Hence, the solution of (7) may be had, in principle, to any desired degree of accuracy. If we, moreover, assume for simplicity that the series of kernels

$$L(\rho|\rho') = \sum_i K_i(\rho|\rho') \tag{14}$$

also converges uniformly, the expression (13) can be put in a concise form:

$$\Pi(\rho) = \Lambda(\rho) + \int d\rho' L(\rho|\rho') \Lambda(\rho') \, . \tag{15}$$

L is called the reciprocal kernel.

A glance at (8) shows that if $\mathbf{b}(\mathbf{r},0) = 0$ then $\Pi(\mathbf{r},t) = 0$, and from (6) there follows $\mathbf{b}(\mathbf{r},t) = 0$. The physical requirement that $\mathbf{B}$ remain zero if initially zero is met by our solutions, and this is true of div $\mathbf{B}$ as well:

$$\text{if } \nabla \cdot \mathbf{B}(\mathbf{r},0) = 0 \text{ then } \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0. \tag{16}$$

B. Condition (3) is not in general satisfied by solutions $\hat{\mathbf{B}}$ of equation (1); at least it is not necessary to impose (3) on the quantity $\hat{\mathbf{B}}$ appearing in the partial integral (3-2). As a result the non-homogeneous term in (4) must be augmented by an additional term from (2), giving

$$\frac{\partial}{\partial t}(\nabla\cdot\hat{\mathbf{B}}) + \mathbf{v}\cdot\nabla(\nabla\cdot\hat{\mathbf{B}}) + \nabla\cdot(\mathbf{v}\cdot\nabla)\hat{\mathbf{B}} = \nu\nabla^2(\nabla\cdot\hat{\mathbf{B}}). \tag{17}$$

Repeating the preceding steps for this equation shows that if $(\nabla\cdot\hat{\mathbf{B}})_{t=0} = 0$, then, nevertheless, $\nabla\cdot(\mathbf{v}\cdot\nabla)\hat{\mathbf{B}}$ develops into a source of $\nabla\cdot\hat{\mathbf{B}}$, even if $[(\mathbf{v}\cdot\nabla)\mathbf{B}]_{t=0}=0$.

Solutions of the substantial diffusion equation are divergence-free if and only if $\nabla\cdot(\mathbf{v}\cdot\nabla)\mathbf{B} = 0$.

For the partial integral under the circumstances of sections 3d and 3c, however, it is not essential that $\nabla\cdot\hat{\mathbf{B}} = 0$, but rather that

$$\nabla^{\circ}\cdot\hat{\mathbf{B}} = 0, \tag{18}$$

by virtue of (3-13). Consider the substantial diffusion equation (1) in the form

$$\frac{d\hat{\mathbf{B}}}{dt} = \nu\nabla^2\hat{\mathbf{B}}\,. \tag{19}$$

If $\hat{\mathbf{B}}$ is expressed in Lagrangian terms, $\hat{\mathbf{B}}\left[\mathbf{r}(\mathbf{r}^{\circ},t),t\right] = \hat{\mathbf{B}}(\mathbf{r}^{\circ},t)$, then (19) may be written

$$\frac{\partial\hat{\mathbf{B}}}{\partial t} = \nu\nabla^2\hat{\mathbf{B}}\,. \tag{19^1}$$

$\partial/\partial t$ and ∇° non commute and

$$\frac{\partial}{\partial t}(\nabla^{\circ}\cdot\hat{\mathbf{B}}) = \nu\,\nabla^{\circ}\cdot\nabla^2\hat{\mathbf{B}}\,. \tag{20}$$

To analyze this equation the mixed derivatives on the right must be replaced by derivatives with respect to the Lagrangian coordinates, $\mathbf{r}^{\circ}$. $\nabla^2\hat{\mathbf{B}}$ may be expressed as

$$\frac{\partial^2 \hat{B}_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial x_k^o}{\partial x_j} \frac{\partial \hat{B}_i}{\partial x_k^o}\right)$$

$$= \frac{\partial x_k^o}{\partial x_j} \frac{\partial^2 \hat{B}_i}{\partial x_j \partial x_k^o} + \frac{\partial^2 x_k^o}{\partial x_j \partial x_j} \frac{\partial \hat{B}_i}{\partial x_k^o} \tag{21}$$

$$= \frac{\partial x_k^o}{\partial x_j} \frac{\partial x_l^o}{\partial x_j} \frac{\partial^2 \hat{B}_i}{\partial x_l^o \partial x_k^o} + \frac{\partial^2 x_k^o}{\partial x_j \partial k_j} \frac{\partial \hat{B}_i}{\partial x_k^o}$$

We consider first the linear velocity shear of section 3c. The Lagrangian trajectories are

$$x_1 = x_1^o \,, \; x_2 = x_2^o + (a x_3^o + b)t \,, \; x_3 = x_3^o . \tag{22}$$

after (3-18) and (3-26); thus,

$$\nabla^2 r^o = \nabla^{o2} r = 0 \,, \tag{23}$$

which reduces (21) to

$$\frac{\partial^2 \hat{B}_i}{\partial x_j \partial x_j} = (\nabla x_k^o \cdot \nabla x_\ell^o) \frac{\partial^2 \hat{B}_i}{\partial x_k^o \partial x_\ell^o} \tag{24}$$

For $\nabla^o \cdot \nabla^2 \hat{\mathcal{B}}$ we have

$$\frac{\partial^3 \hat{B}_i}{\partial x_i^o \partial x_j \partial x_j} = (\nabla x_k^o \cdot \nabla x_\ell^o) \frac{\partial^3 \hat{B}_i}{\partial x_k^o \partial x_\ell^o \partial x_i^o} \tag{25}$$

$$+ \frac{\partial^2 \hat{B}_i}{\partial x_k^o \partial x_\ell^o} \frac{\partial}{\partial x_i^o} (\nabla x_k^o \cdot \nabla x_\ell^o) \,.$$

The following relations are easily obtained from (22)

$$\nabla x_1^o = \boldsymbol{e}_1, \quad \nabla x_2^o = \boldsymbol{e}_2 - at\,\boldsymbol{e}_3, \quad \nabla x_3^o = \boldsymbol{e}_3 \;; \tag{26}$$

$$\left.\begin{aligned} \nabla x_1^o \cdot \nabla x_1^o &= \nabla x_3^o \cdot \nabla x_3^o = 1 \\ \nabla x_1^o \cdot \nabla x_2^o &= \nabla x_1^o \cdot \nabla x_3^o = 0 \\ \nabla x_2^o \cdot \nabla x_2^o &= 1 + a^2t^2 \\ \nabla x_2^o \cdot \nabla x_3^o &= -\,at. \end{aligned}\right\} \qquad (27)$$

The last term in (25) vanishes, leaving

$$\nabla^o \cdot \nabla^2 \hat{\mathbf{B}} = \nabla^{o^2} (\nabla^o \cdot \hat{\mathbf{B}}) - 2\,at \frac{\partial^2}{\partial x_2^o \partial x_3^o} (\nabla^o \cdot \hat{\mathbf{B}}) + a^2t^2 \frac{\partial^2}{\partial (x_2^o)^2} (\nabla^o \cdot \hat{\mathbf{B}}), \qquad (28)$$

and (20) becomes

$$\nu^{-1} \frac{\partial}{\partial t}(\nabla^o \cdot \hat{\mathbf{B}}) = \nabla^{o^2}(\nabla^o \cdot \hat{\mathbf{B}}) - 2\,at \frac{\partial^2}{\partial x_2^o \partial x_3^o} (\nabla^o \cdot \hat{\mathbf{B}}) + a^2t^2 \frac{\partial^2}{\partial (x_2^o)^2} (\nabla^o \cdot \hat{\mathbf{B}}). \qquad (29)$$

Note that in general $\partial(\nabla^o \cdot \hat{\mathbf{B}})/\partial t \neq 0$; however, when $t = 0$ the equation is momentarily

$$\frac{\partial}{\partial t}(\nabla^o \cdot \hat{\mathbf{B}}) = \nu\, \nabla^{o^2}(\nabla^o \cdot \hat{\mathbf{B}}).$$

As before, we see that if $\nabla^o \cdot \mathbf{B} = 0$ initially, there is no tendency for sources to develop, and we surmise that $\nabla^o \cdot \hat{\mathbf{B}}$ remains thereafter zero. This inference is given added weight by considering what happens when t is small but not zero. The last term on the right of (29) can be ignored compared with the first, and for $t \rightarrow 0$,

$$\nabla^{o2}(\nabla^o\cdot\hat{\mathbf{B}}) - \nu^{-1}\frac{\partial}{\partial t}(\nabla^o\cdot\hat{\mathbf{B}}) = 2at\frac{\partial^2}{\partial x_2^o\partial x_3^o}(\nabla^o\cdot\hat{\mathbf{B}}). \tag{30}$$

For the sake of brevity we define the operators

$$\partial^2 = \frac{\partial^2}{\partial x_2^o\partial x_3^o}\ ,\quad \partial'^2 = \frac{\partial^2}{\partial x_2^{o\prime}\partial x_3^{o\prime}}$$

and let ρ designate the combination of variables $(\mathbf{r}^o, t)$ while

$$D(\rho) = \nabla^o\cdot\hat{\mathbf{B}}\,.$$

Equation (3) in this symbolism becomes

$$\nabla^{o2}D - \nu^{-1}\frac{\partial D}{\partial t} = 2\,a\,t\partial^2 D. \tag{31}$$

Comparing this with (4-1) and (4-14) we see that

$$D(\rho) = \int d\rho'\,G(\rho|\rho')2at\partial'^2 D(\rho') + \nu^{-1}\int d^3\mathbf{r}^{o\prime}(GD')_{t=0}\,. \tag{32}$$

Operating on this with with ∂^2 gives an integral equation for $\partial^2 D$, viz.

$$\partial^2 D(\rho) = \varphi(\rho) + \int d\rho'\,2at\partial^2 G(\rho|\rho')\partial'^2 D(\rho'), \tag{33}$$

where

$$\varphi(\rho) = \nu^{-1}\int d^3\mathbf{r}^{o\prime}\left[(\partial^2 G)D'\right]_{t=0}\,. \tag{34}$$

It has a solution of the form

$$\partial^2 D(\rho) = \varphi(\rho) + \int d\rho'\,M(\rho|\rho')\varphi(\rho'), \tag{35}$$

after (IV-7) and (IV-15), M being the kernel reciprocal to $2at\partial^2 G$. As before, if

$$D_{t=0} = \nabla^o\cdot\hat{\mathbf{B}}(\mathbf{r}^o,0) = 0 \tag{36}$$

then $\varphi(\rho) = 0$ by (34), and consequently $\partial^2 D = D = 0$ by

(35) and (32).

Interpreting this result is not quite so simple as before. D is expressed in Lagrangian terms, so the partial time derivative in (31) is equivalent to a substantial derivative; accordingly, what we have shown is that if $D(\mathbf{r}^o, 0) = 0$ then, subsequently, $D(\mathbf{r}^o, t) = 0$ at a particle <u>moving with the fluid.</u> But, since this must be true for all particles, we conclude that

$$\nabla^o \cdot \hat{\mathbf{B}}(\mathbf{r}^o, t) = 0 \text{ everywhere in the fluid and at all times provided only } \nabla^o \cdot \hat{\mathbf{B}}(\mathbf{r}^c, 0) = 0 \tag{37}$$

Equation (3-14) follows at once.

As regards the approximate solutions of section 3d, the last terms of (21) and (25) are no longer zero. Nevertheless, since $\mathbf{v}$ is slowly varying, the curvature of the trajectories, of which the second derivatives of the Lagrangian coordinates in (21) and (25) are a measure, is very small, and the last terms in these equations are negligible compared with the first terms and may be dropped, leading once again to (29).

C. It remains to treat the divergence criteria for the solutions of section 3c. The essential steps are exactly the same as in A of this appendix. Taking the divergence of (4-18b) shows that

$$\nabla^2(\nabla \cdot \mathbf{F}) + k^2 \nabla \cdot \mathbf{F} = \nu^{-1} \mathbf{v} \cdot \nabla(\nabla \cdot \mathbf{F}) \tag{38}$$

provided

$$\nabla \cdot (\mathbf{F} \cdot \nabla)\mathbf{v} = 0 . \tag{39}$$

Thus we let $\mathfrak{f}$ represent either $\mathbf{F}$ or $\nabla \cdot \mathbf{F}$ and $\check{\mathfrak{f}}$ either $\check{\mathbf{F}}$ or $\nabla \cdot \check{\mathbf{F}}$. Then from (4-24) and (4-25),

$$\mathfrak{f} = \check{\mathfrak{f}} - \nu^{-1} \int d^3\mathbf{r}'\, G(\mathbf{v}' \cdot \nabla')\mathfrak{f}' \tag{40}$$

and

$$(\mathbf{v} \cdot \nabla)\mathfrak{f} = (\mathbf{v} \cdot \nabla)\check{\mathfrak{f}} - \nu^{-1} \int d^3\mathbf{r}'\, (\mathbf{v} \cdot \nabla G)(\mathbf{v}' \cdot \nabla')\mathfrak{f}', \tag{41}$$

which have precisely the same forms as (6) and (7), respectively. We may state at once that if $\check{f} = 0$ then both $(\vee \cdot \nabla) f$ and f vanish. Once a particular solution of the homogeneous equation is selected, $\mathbf{F}$ is given by (40), and this fixes the form of $\nabla \cdot \mathbf{F}$: it is that for which the "inhomogeneous" term is $\nabla \cdot \check{\mathbf{F}}$. And

$$\text{if } \nabla \cdot \check{\mathbf{F}} = 0 \text{ then } \nabla \cdot \mathbf{F} = 0. \tag{42}$$

This leads to equation (4-30).

D. From the physical and mathematical analyses of the "inhomogeneous" differential equations studied in this appendix, we hazard a generalization. Consider an equation of the form

$$\nabla^2 A - a^2 \frac{\partial A}{\partial t} = \mathcal{L} A,$$

where $\mathbf{A}(\mathbf{r},t)$ is an arbitrary vectory function and $\mathcal{L}$ denotes an operator linear in the space derivatives. If $\mathbf{A}(\mathbf{r},0) = 0$ we assert that since $\mathcal{L} A(\mathbf{r},0) = 0$, a result corresponding to (15), there are initially no sources for $\mathbf{A}$, and moreover, because of this there is no tendency to generate sources. Whenever the latent "sources," $-\mathcal{L} A$, depend on A in such a way that they are zero when A is everywhere zero, they are given no chance to develop, and the quantity A, once zero remains zero thereafter.

References

Brand, L., 1947. Vector and Tensor Analysis (New York: Wiley), p. 263.

Courant, R., and Hilbert, D., 1953. Methods of Mathematical Physics, Vol. I. (New York: Interscience).

Elsasser, W., 1954. Phys. Rev., 95, 1.

Inglis, D., 1955. Rev. Mod. Phys., 27, 212.

Lundquist, S., 1951. Phys. Rev., 83, 307.
1952. Arkiv Fysik, 5, 297.

Morse, P., and Feshbach, H., 1953. Methods of Theoretical Physics, Vols, I and II. (New York: McGraw-Hill).

Page, C., 1955. Physical Mathematics. (Princeton: Van Nostrand).

Skabelund, D., 1955. Technical Report 17 of this series. University of Utah, Salt Lake City.

Sommerfeld, A., 1949. Partial Differential Equations in Physics. (New York: Academic Press).

Whittaker, E., and Watson, J., 1920. Modern Analysis, 3d. Ed. (Cambridge Univ. Press).

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